Physical Models of the Toroidal Dipole

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Motivation

Figure 1. James Clerk Maxwell

We took our research topic directly from the Father of Electrodynamics (Maxwell, 1861): “Let there be a circular ring of uniform section, lapped uniformly with covered wire. It may be shown that if an electric current is passed through this wire, a magnet placed within the coil of wire will be strongly affected, but no magnetic effect will be produced on any external point. “The effect will be that of a magnet bent round till its two poles are in contact”.

Introduction

The toroidal solenoid is a unique object exhibiting many interesting properties and very rich physics, generally not very well known, besides what we read in the college physics textbooks. The current flowing in its winding is characterized by a new electromagnetic property called toroidal dipole, or anapole (pole-less) moment (Dubovik et al, 1989). These dipoles recently became a subject of an extensive theoretical and experimental study. But the standard textbook presentations of the multipole expansions of electromagnetic fields are usually incomplete, since they are valid only in the regions outside of localized currents. As a result, they miss the existence of, and do not even mention the toroidal moments.

Everybody knows the properties of a bar magnet. Its strength, called the magnetic dipole moment $\mu$, is characterized by its response to an external magnetic
field, measured by the torque of the field on the magnet. Another known realization of a magnetic dipole is a loop of electric current.

Figure 2.

The well-known properties of the two dipoles are summarized below: in an external field they experience a torque $\tau$ and acquire an additional energy $U_m$.

Figure 3.

Magnetic Dipole

- Moment
  \[ \vec{\mu} = i \Delta \vec{m} \]
- Properties in $B$ field:
  \[ \tau = \vec{\mu} \times \vec{B} \]
  \[ U_m = -\vec{\mu} \cdot \vec{B} \]

The exact definitions of the magnetic dipole moments in terms of the current density $j$ for currents in wires and the magnetization $M$ (magnetic dipole moment per unit volume) for permanent magnets are shown below (Wangsness, 1979).

Figure 4.

Magnetic Dipole Definition

- In terms of current density vector:
  \[ \vec{i} = \frac{\vec{I}}{\alpha} \quad \vec{\mu} = \frac{1}{2} \int \vec{\tau} \times \vec{\mu} dV \quad \vec{\mu} = j \mu dA \]
- In terms of magnetization $M$:
  \[ \vec{\mu} = \int M dV \]
\( \mathbf{i} \) is the current in the loop, \( a \) is the cross-sectional area of the wire, and \( A \) is the area surrounded by the loop, \( dV \) is the volume element, and \( \mathbf{r} \) is a position vector of a point inside the wiring for the electric case or inside a magnet for the magnetized material.

**Toroidal systems**

There is one more elementary magnetic system, frequently overlooked – the toroidal dipole. If we follow the suggestion of the Father of Electrodynamics and bend a bar magnet into shape of a donut, thus joining its ends, with a north pole coming in contact with the south pole – then its entire magnetic field, represented by the field lines on Figure 2 above will be concentrated inside, and no field will be left even in the hole of the torus. Such a magnet has no magnetic poles. Red magnetic lines below are totally confined within the torus, with the blue “bound” or “magnetization” currents of the atomic electrons flowing “poloidally” (along the polar lines) on the surface of the magnet.

Figure 5.

The question to ask is: would this toroid experience any force, if placed in the external magnetic or electric field, even if it does not produce any external field itself? This paper is devoted to answering this question.
A similar electric system is formed by bending a straight solenoid connected to a DC power supply into a torus, and observing the same effect that this electric toroid also confines its magnetic field to its interior.

Figure 6.

It was shown (Dubovik, 1989), that both toroids will interact with an external (coming from sources outside of them) magnetic field $B$ through their toroidal magnetic dipole moments with following formulas for the electric or magnetic toroids:

Figure 7.

**Toroidal Dipole Definition**

- Toroidal magnetic dipole moment:
  \[
  \bar{t}_m = \frac{1}{6} \int \bar{r} \times (\bar{r} \times \bar{j}) dV
  \]
- For a magnetized material:
  \[
  \bar{t}_m = \frac{1}{2} \int \bar{r} \times (\bar{M}) dV
  \]

Magnetic dipole moment $\mu$ of a small thin current loop can be calculated from a textbook formula $\mu = iA = jaA$, where $j$ is the current density, $i$ is the current, $a$ is the cross-sectional area of the wire (and so $i = ja$) and $A$ is the area surrounded by the loop.
Figure 8. Red arrows are magnetization vectors $\mathbf{M}$, $\mathbf{j}$ is the surface current density, and the green vector $\mathbf{t}_m$ is the toroidal dipole moment directed along the torus’ symmetry axis.

By using an exact mathematical analogy between a current loop and the magnetization loop of the toroid, with $\mathbf{j}$ replaced by $\mathbf{M}$, the toroidal dipole moment $\mathbf{t}_m$ for a thin small toroid should be equal approximately $\mathbf{t}_m = \mathbf{MaA}$, where $a$ is the cross-sectional area of the torus (inside a blue loop), and $A$ is the area surrounded by the toroid, if we count its radius to the central line of the torus (where the red arrows are). In our toroid the conditions of a small thin toroid ($a \ll A$) are not satisfied, so that we do not have a reliable theoretical formula for $\mathbf{t}_m$.

Figure 9. Comparison of two toroids

Current-generated, small (point) toroid with the current density $\mathbf{j}$ has a toroidal dipole moment $\mathbf{t}_m$ and interacts with the external field $\mathbf{B}$ via the following torque $\tau$ and potential energy $U_m$ (Dubovik, 1989):
Our experimental magnetic toroid with the approximately azimuthal (directed around the loop) magnetization \( \mathbf{M} \) has the toroidal dipole moment \( \mathbf{t}_m \) given below:

\[
\mathbf{t}_m = \frac{1}{2} \int \vec{r} \times (\vec{r} \times \vec{M}) \, dV
\]

Figure 11.

and the same point-like interactions for torque and energy.

As we see, toroidal dipole moments interact with the curl of the external magnetic field’s curl (curl \( \mathbf{B} \)) in the same way, in which magnetic dipole moments interact with an external magnetic field: they feel a torque \( \mathbf{\tau} \), giving them potential energy \( U_m \).

They interact then not directly with the magnetic field, but with its curl \( \mathbf{B} = \nabla \times \mathbf{B} \). Curl is a derivative operation on the components of vector \( \mathbf{B} \), producing another related vector describing the circulation of vector \( \mathbf{B} \) around a given point. Intuitively, the curl of a vector field measures the tendency for the vector field to swirl around. Imagine that the vector field represents the velocity vectors of water in a lake. If the velocity swirls around, then when we stick a paddle wheel into the water, it will tend to spin.

We do not need to go into the mathematical detail of the curl of \( \mathbf{B} \), because the Ampere-Maxwell Law of Electrodynamics tells us what it is at every point:
Figure 12.

\[ \nabla \times \vec{B} = \mu_0 \vec{j} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \]

In introductory physics courses this Ampere’s Law is usually studied in its equivalent integral form (c being the speed of light), and \( \varepsilon_0 \) and \( \mu_0 \) are electric permittivity and magnetic permeabilities of the free space, respectively.

Figure 13.

\[ \oint \vec{B} \cdot d\vec{s} = \mu_0 i + \frac{1}{c^2} \frac{\partial}{\partial t} \int \vec{E} \cdot d\vec{A} \]

Both forms tell us, that curl \( \vec{B} \) is non-zero at points where we have a flow of current density \( \vec{j} \), or an electric field \( \vec{E} \) changing in time. So, we may expect, that a toroid will experience a torque and an energy change either from a current flowing through its hole, or from a time-dependent electric field inside the hole. A curl-less (for example, uniform) magnetic field should not act on the toroid, only a direct contact with the conduction current or a “displacement current”, which is the second term on the right hand sides of both equations. The vector (derivative) form of the Ampere-Maxwell Law informs us that our toroid can interact with the conduction current density \( \vec{j} \), or with the time-dependent displacement current density, if they have a non-vanishing component along the axis of symmetry of the toroid.

**Laboratory model of a magnetic toroid**

Figure 14. Two realizations of a toroidal dipole in our experiments
Our toroidal magnet has a small deviation from the strict ideal toroid - for practical reasons (it is not easy to magnetize a neodymium ring strictly circumferentially), it was constructed from 12 magnetic dipole arc, or ‘pie’, segments. When put in a ring a chain of magnetic dipoles back to back, with N-pole of one touching the S-pole of the next one, the magnets stick to each other with a force of about 70 lb. Each segment is magnetized perpendicularly to its middle radial line. Because of this, its magnetization is not strictly azimuthal, thus the torus has a net *dodecapole* magnetic moment with a very small and very short-ranged external magnetic field of the 12-pole polarity. The magnetic dipole moments of all segments add to a net vector of zero, of course. We made our magnetic toroid from magnetized neodymium NdFeB grade 52 segments with inner radius 1”, outer radius 2”, vertex angle 30°, and square cross section with an area 1”x1”.

Figure 15. Red arrows show directions of magnetic dipole moments creating the toroidal moment
Magnets have 1.48 Tesla residual internal flux density (magnetic field) $B$, as measured in the gap (black insert on the toroid’s picture) left in the toroid. Its uniform magnetization is $M = B / \mu_0 = 1.18 \times 10^6$ A/m, close to circumferential. Surface bound current density is $K = M = 1.18 \times 10^6$ A/m, total equivalent magnetization current going through the hole of the torus is $2.82 \times 10^5$ A. Mass of the toroid is 1.184 kg, moment of inertia calculated about its big diameter: $I = 1.01 \times 10^{-3}$ kg·m$^2$.

Figure 16.
Arrows show ideal (on left) vs actual (on right) magnetization directions:

Figure 17. Shown below is measured radial field leaking on the ring’s outside rim because of the sudden changes of the magnetization direction: up to 50-80 mT on the magnet’s surface, which is 5% of the internal B-field, together with a weak magnetic field leakage from the side of the toroid as visualized with iron filings:

Discontinuous magnetization directions introduce dodecapole (12-pole) moment to our torus in addition to toroidal dipole moment. We checked that its escaping magnetic field range is up to 2 cm on both rims of the torus, and so it will not practically interact with a current passing through its center.
Figure 18. Radial component of magnetic field escaping from the outside rim of the ring

![Graph of Radial Component](image1)

Figure 19.

![Graph of Tangential Escaping Field vs Distance](image2)

Our toroid is not expected to react to a uniform, curl-less magnetic field; and yet, it actually feels a strong torque, when placed in a uniform field inside Helmholtz coils (below). This is a demonstration of the dodecapole component of our toroid.

Figure 20.
Experiments with magnetic toroid

The formula for torque $\tau$ and energy $U_m$ and the Ampere-Maxwell Law suggest two experiments:

**Experiment 1:** immersing the toroid in an electrolyte with a volume DC current density $j$ to measure its torque on the toroid and so find the toroid moment.

**Experiment 2:** immersing the toroid in time-dependent electric field $E$, and finding the torque of the displacement current on the toroid.

Experimenting within an electrolyte is not practical. We cannot create a uniform current density everywhere inside the toroid. Instead we did a variation of Experiment 1: placing the toroid near linear currents and seeing what happens. Currents are carrying inside them the curl $B$ represented by the current density. This density must overlap with the toroid, if there is to be an interaction. We assume that the toroid’s magnetization is not changed by the external magnetic fields we apply, which are by a factor of $10^5$ smaller than the inner field in the toroid. We checked the torque for two situations – when the current is passing inside the toroid’s hole or outside of it.

Effective interaction between toroid and an electric current

For a point-like toroid encircling a current density $j$ the theory predicts the torque $\tau = t \times \mu_0 j$. It cannot be easily directly generalized to the torque on the physical toroid from a current carried by the wire, because it was derived under the assumption that a uniform current density is present everywhere inside the toroid. For a physical toroid encompassing wire with a DC current $i$ at a large distance from the toroid’s magnets we will use an empirical torque formula with a toroidal constant $t$ and a vector $t = t_n$. Here $n$ is the unit vector of the toroidal moment, and so $\tau = t \times i$, with the magnitude $\tau = t i \sin \vartheta$, where $\vartheta$ is the angle between $n$ and the direction of current. This torque tends to rotate the toroid’s dipole moment into alignment with the current vector.

Figure 21.
The actual torque on toroid’s dipoles is created by the wire’s magnetic field. This is intuitively to be expected: individual circumferential magnetic dipoles try to align with the wire’s circular magnetic lines. It is easy to notice that in a toroid’s equilibrium position the magnetization loops will be overlapping with the circular magnetic lines of the current, minimizing the toroid’s magnetic energy.

It may be also said that the toroidal dipole moment interacts in effect with the curl $B$, which is produced inside the wire – and so inside the hole of the toroid – by the current density, if the wire passes inside the toroid.

Instead of trying to measure directly a very small torque, we use an old torsional pendulum method first exploited in 1883 by Carl Friedrich Gauss (of the gauss unit fame) to measure the Earth’s magnetic field. Interaction of the toroid with an electric current can be measured from the torsional oscillations of torus suspended near the current-carrying wire on a monofilament fishing line, with torsion constant $k$.

We confirmed a contrasting behavior of a toroidal dipole and of a magnetic dipole. The ring on the right, below, is magnetized perpendicular to its plane and is then a magnetic dipole oriented along ring’s axis. It aligns itself with the magnetic field perpendicular to the wire, while toroid’s axis aligns with the current direction.

Figure 22.

Theory

Passing a wire (with DC current) inside and outside of the magnetic toroid, we recorded frequency of toroid’s oscillations as a function of the current.
In torsional oscillations of a torus on a string, with current $i$ passing outside of its hole, torque should be mechanical only, provided by the string. If external current $i$ passes outside the toroid’s hole, no interaction with it is predicted, and only mechanical torque due to a string with a torsion coefficient $k$: $\tau_{\text{mechanical}} = -k\theta$ is present. If the current passes through toroid, both mechanical and electromagnetic torques will act. Torque due to interaction of toroidal moment with current $i$:

$$\tau = -t \cdot i \cdot \sin \theta = -t \cdot i \cdot \theta \quad \text{(for small angles)}$$

Modified period of our torsion pendulum in terms of the toroid constant $t$, when both mechanical and magnetic torques are present, is:

\[ T = 2\pi \sqrt{\frac{I}{It+k}} \]

\[ f^2 = \left( \frac{t}{4\pi^2 I} \right) i + \frac{k}{4\pi^2 I} \]

where $I$ is the moment of inertia of the toroid around its diameter.

**Results**

Initial measurements of the magnetic toroid’s oscillations, with current wire in and out of its hole, demonstrated that when wire passes outside, the varying current has no effect on toroid (red points), as expected, because there $\text{curl } B = 0$, $B$ is curl-less. When the wire passes through the toroid, we expected an increase in the frequency with increasing current, which is confirmed below (blue points):

**Figure 24.** $T =$ period of oscillations, $I =$ moment of inertia of the toroid around its diameter, $i =$ external current, $t =$ toroid constant, $k =$ torsion coefficient, $f =$ frequency of oscillations.
To find the toroidal constant $t$, which can be thought of as “an effective toroidal moment” for our experiment, equal to the amount of torque per one ampere of current, we did several series of measurements of the frequency of oscillations of our freely suspended toroid as a function of the current $i$ in a wire passing through the opening of the torus.

From a typical graph of $f^2 = T^{-2}$ vs $i$ we can extract toroidal and torsion constants: from the slope $= 2.97 \times 10^{-4}$ 1/As we get $t = (4\pi I) (\text{slope}) = 1.18 \times 10^{-5}$ Nm/A, and from the intercept $k = 4.66 \times 10^{-5}$ Nm.

The important numerical result of this experiment: the torque on toroid from 1 A current passing through its hole equals $t = 1.18 \times 10^{-5}$ Nm/A.

The torque amplitude at 7 A in our experiments was only 0.0826 milli-Nm. Our calculated from this experiment toroidal dipole moment, when using the current density averaged over area $A$ of the torus, $J_{av} = i/A$, turns out to be: $\tau = t_m \mu_0 (i/A)$ leads to $t_m = tA/\mu_0 = 4.28 \times 10^{-2}$ Am$^3$.

**Experiment 2**

We unsuccessfully tried to observe a torque on the toroid placed inside a capacitor with a sudden drop of the electric field in it, which was supposed to result
in a short impulse of torque of the collapsing displacement current acting on the toroid.

Interaction with displacement current had no measurable or even observable effect. We estimated that the rate of change of the electric field collapsing by $\Delta E$ in time $t_c$ was

$$\frac{\Delta E}{t_c} = 4.44 \times 10^{14} \text{ V/s}$$

so the displacement current density was

$$\varepsilon_0 \frac{\Delta E}{t_c} = 3.93 \times 10^3 \text{ A/m}^2$$

to be compared with the current density in the wire

$$j = \frac{i}{a} = 6.32 \times 10^6 \text{ A/m}^2.$$ The average torque on toroid (using the theoretical $t_m$) was expected to be

$$\tau = t_m \mu_0 \varepsilon_0 \frac{\Delta E}{t_c} = \mu_0 (MaA) (\varepsilon_0 \frac{\Delta E}{t_c}) = 4\pi 10^{-7} \text{ (Tm/A)} (2.41 \text{ Am}^3) (3.93 \times 10^3 \text{ A/m}^2) = 1.19 \times 10^{-2} \text{ Nm, very small.}$$

**Electric toroid**

We tried to measure the toroidal moment of the DC toroidal coil. We do not have any results here, because of several experimental difficulties.
Experimental difficulties with the electric toroid:

- Toroid’s powering and winding wires would overheat
- Extra torque applied by powering wires
- Iron core was magnetized
- Its winding has a pitch angle, giving the coil a net magnetic dipole moment from its net circumferential current. This magnetic dipole presence obscured toroidal interactions.
- Powering toroidal coil from attached to it batteries has increased its inertia, making it less responsive to the torque

Possible future experiments

Toroids have many other peculiar properties. In vacuum, the magnetic field outside the toroidal dipole is zero. But fields do appear outside the dipole in an electrodynamic medium with permittivity $\varepsilon_0$ and permeability $\mu_0$; they are
proportional to $\varepsilon_0\mu_0^{-1}$, and arise because of the fact that, according to the macroscopic electrodynamics of a homogeneous medium, the medium should be considered as permeating the dipole itself (Ginsburg, 1989). We plan to imbed the magnetic toroid in a medium of known $\varepsilon_0$ and $\mu_0$ and try to detect this outside field.

Toroidal magnet as an electric-dipole antenna: if we rotate our magnetic toroid around its diameter, we could observe the theoretically predicted effect of the magnetic field (which is normally locked inside a toroid at rest) escaping outside of the rotating toroid, as a part of the quasi-static, or radiation, field of the electric-dipole type. An oscillating electric and magnetic field will be generated around a fixed conducting loop placed near the rotating magnetic toroid (Fernandez-Corbaton, 2017). Rotating toroid should display the same radiated fields as an electric toroid with AC current.

Figure 28. Toroid on a rotator

Toroids in fundamental physics

Figure 29.
Atoms and nuclei were found to have toroidal dipole moments because of presence of a parity-nonconserving (non-mirror symmetric) weak nuclear force (Wood et al, 1997). For example, while the normal spherically symmetric hydrogen atom has its electron moving along a classical circular orbit under the attraction of the central electric force mediated by virtual photons, in the presence of the weak nuclear force mediated by the Z^0 bosons its path becomes slightly helical, spiraling around the circular coulombic orbit. Such an atom is then a miniature toroid and has corresponding toroidal, or anapole, moment.

Anapole dark matter theory: physicists propose (Ho C. et al, 2013) that dark matter, an invisible form of matter that makes up 26 percent of all the matter in the universe, may be made out of a type of basic particle possessing a rare, donut-shaped magnetic field, called an anapole.

Applications of toroidal magnets in technology

Research in the last few years has demonstrated, that static toroidal moments exists in various materials both microscopically and macroscopically, covering transition metal ions, biological and chemical macromolecules, bulk crystals and glasses (Talebi et al, 2017).

Figure 30. Miniature toroids are used in artificially engineered electromagnetic media - toroidal metamaterials. Examples:

Ring-shaped molecular toroidal magnets on the left and a toroidal metamaterial with a unit cell consisting of four connected square copper wire loops, imbedded in a dielectric, in the middle. Their toroidal grainy inside magnetization is called macroscopic toroidization (density of toroidal dipoles), on the right. Unusual properties of toroidal matter opened a horizon of potential applications in data
storage disks and in designing of low loss metamaterials or metadevices, such as
ultrasensitive sensors and diagnostic tools. (Ye et al, 2013), (Zhao et al, 2016).

Conclusions:

We have built and investigated a physical model of the third (in addition to
electric and magnetic dipoles) elementary electromagnetic dipole – the magnetic
toroid:

Figure 31.

We believe that our paper is the first in the physics literature investigating the
properties of a laboratory model of a magnetic toroid.

We confirmed qualitatively that the permanent-magnet toroid interacts only with
the curl of the magnetic field, and not with the field itself. It does not exhibit any
noticeable interaction with a curl-less field outside of the linear current, in
agreement with theoretical expectations.

We measured the toroidal dipole moment. Both lack of adequate theory and a non-
perfect nature of our toroid prevent a meaningful comparison of theory and
experiment.

Our toroid is acting as a torsional curl meter: while magnetic moments of wire loops
are used in simple gauss meters to measure the magnetic field, toroids may be used
as “curl \( \mathbf{B} \) meters” to detect the curl of the magnetic field. They also may act as
ammeters for live wires passing in their holes, measuring currents, or as detectors
of the variable electric fields.

Toroidal conduction currents or magnetic loops represent the simplest of the
possible multipolar, localized currents or magnetizations that produce only
“contact”, finite-range magnetic field distributions. These are not included in the
usual multipole expansions describing the field outside of the sources. The simplest toroidal multipole is a dipole.

Figure 32. What if the Earth was toroidal? (-:

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Figure 5:

Figure 30:

Figure 31:

Figure 32: